

HYERS-ULAM STABILITY OF JENSEN FUNCTIONAL EQUATION ON AMENABLE SEMIGROUPS

BOUIKHALENE BELAID AND ELQORACHI ELHOUCIEN

ABSTRACT. In this paper, we give a proof of the Hyers-Ulam stability of the Jensen functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x), \quad x, y \in G,$$

where G is an amenable semigroup and σ is an involution of G .

1. INTRODUCTION

In 1940, Ulam [17] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Given a group G_1 , a metric group (G_2, d) , a number $\epsilon > 0$ and a mapping $f: G_1 \rightarrow G_2$ which satisfies $d(f(xy), f(x)f(y)) < \epsilon$ for all $x, y \in G_1$, does there exist an homomorphism $g: G_1 \rightarrow G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that $d(f(x), g(x)) < k\epsilon$ for all $x \in G_1$?

In 1941, Hyers [10] considered the case of approximately additive mappings under the assumption that G_1 and G_2 are Banach spaces.

Rassias [16] provided a generalization of the Hyers' Theorem for linear mappings, by allowing the Cauchy difference to be unbounded.

Beginning around the year 1980, several results for the Hyers-Ulam-Rassias stability of very many functional equations have been proved by several researchers. For more detailed, we can refer for example to [5], [7], [8], [11], [12], [15], [19].

Let G be a semigroup with neutral element e . Let σ be an involution of G , which means that $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$.

We say that $f: G \rightarrow \mathbb{C}$ satisfies the Jensen functional equation if

$$(1.1) \quad f(xy) + f(x\sigma(y)) = 2f(x)$$

for all $x, y \in G$.

The Jensen functional equation (1.1) takes the form

$$(1.2) \quad f(xy) + f(xy^{-1}) = 2f(x)$$

for all $x, y \in G$, when $\sigma(x) = x^{-1}$ and G is a group.

The stability in the sens of Hyers-Ulam of the Jensen equations (1.1) and (1.2) has been studied by various authors when G is an abelian group or a vector space. The interested reader can be referred to Jung [13], Kim [14] and Bouikhalene et al. [1, 2].

Recently, Faiziev and Sahoo [4] have proved the Hyers-Ulam stability of equation (1.2) on some noncommutative groups such as metabelian groups and $T(2, K)$, where K is an arbitrary commutative field with characteristic different from two.

In this paper, motivated by the ideas of Forti, Sikorska [6] and Yang [18], we give a proof of the Hyers-Ulam stability of the Jensen functional equation (1.1), under the condition that G is an amenable semigroup.

2. HYERS-ULAM STABILITY OF EQUATION (1.1) ON AMENABLE SEMIGROUPS

In this section we investigate the Hyers-Ulam stability of equation (1.1) on an amenable semigroup G .

We recall that a semigroup G is said to be amenable if there exists an invariant mean on the space of the bounded complex functions defined on G . We refer to [9] for the definition and properties of invariant means.

The main result of the present paper is the following.

Theorem 2.1. *Let G be an amenable semigroup with neutral element e . Let $f: G \rightarrow \mathbb{C}$ be a function satisfying the following inequality*

$$(2.1) \quad |f(xy) + f(x\sigma(y)) - 2f(x)| \leq \delta$$

for all $x, y \in G$ and for some nonnegative δ . Then there exists a unique solution g of the Jensen equation (1.1) such that

$$(2.2) \quad |f(x) - g(x) - f(e)| \leq 3\delta$$

for all $x \in G$.

First, we recall the following lemma which is a generalization of the useful lemma obtained by Forti and Sikorska in [6]. The proof was given by the authors in [3].

Lemma 2.2. *Let G be a semigroup and B be a Banach space. Let $f: G \rightarrow B$ be a function for which there exists a solution g of the Drygas functional equation*

$$(2.3) \quad g(yx) + g(\sigma(y)x) = 2g(x) + g(y) + g(\sigma(y)), \quad x, y \in G$$

such that $\|f(x) - g(x)\| \leq M$, for all $x \in G$ and for some $M \geq 0$.

Then

(2.4)

$$g(x) = \lim_{n \rightarrow +\infty} 2^{-2n} \{ f^e(x^{2^n}) + \frac{1}{2} \sum_{k=1}^n 2^{k-1} [f^e((x^{2^{n-k}} \sigma(x)^{2^{n-k}})^{2^{k-1}}) + f^e((\sigma(x)^{2^{n-k}} x^{2^{n-k}})^{2^{k-1}})] \} \\ + 2^{-n} \{ f^o(x^{2^n}) + \frac{1}{2} \sum_{k=1}^n [f^e((x^{2^{k-1}} \sigma(x)^{2^{k-1}})^{2^{n-k}}) - f^e((\sigma(x)^{2^{k-1}} x^{2^{k-1}})^{2^{n-k}})] \},$$

where $f^e(x) = \frac{f(x) + f(\sigma(x))}{2}$, $f^o(x) = \frac{f(x) - f(\sigma(x))}{2}$ are the even and odd part of f .

Lemma 2.3. Let G be semigroup with neutral element e and B a complex Banach space. Assume that $f: G \rightarrow B$ satisfies the inequality

$$(2.5) \quad |f(xy) + f(x\sigma(y)) - 2f(x)| \leq \delta$$

for all $x, y \in G$ and for some $\delta \geq 0$. Then the limit

$$(2.6) \quad g(x) = \lim_{n \rightarrow \infty} 2^{-n} f(x^{2^n})$$

exists for all $x \in G$ and satisfies

$$(2.7) \quad |f(x) - g(x) - f(e)| \leq \frac{3\delta}{2} \quad \text{and} \quad g(x^2) = 2g(x)$$

for all $x \in G$.

The function g with the condition (2.7) is unique.

Proof. We define a function $h: G \rightarrow \mathbb{C}$ by $h(x) = f(x) - f(e)$. First, by using the inequality (2.5) we obtain

$$(2.8) \quad |h(xy) + h(x\sigma(y)) - 2h(x)| \leq \delta$$

for all $x, y \in G$. Letting $x = e$ in (2.8), we get

$$(2.9) \quad |h^e(y)| \leq \frac{\delta}{2}$$

for all $y \in G$. Similarly, we can put $y = x$ in (2.8) to obtain

$$(2.10) \quad |h(x^2) + h(x\sigma(x)) - 2h(x)| \leq \delta$$

for all $x \in G$. Since $h(x\sigma(x)) = h^e(x\sigma(x))$, so from (2.9) and (2.10), we have

$$(2.11) \quad |h(x^2) - 2h(x)| \leq \frac{3\delta}{2}$$

for all $x \in G$. Now, by applying some approach used in [4], we get the rest of the proof. \square

Proof of Theorem 2.1. In the proof we use some ideas from Yang [18] and Forti, Sikorska [6].

Setting $x = e$ in (2.1) we have

$$(2.12) \quad |f^e(y) - f(e)| \leq \frac{\delta}{2}$$

for all $y \in G$.

The inequalities (2.1), (2.12) and the triangle inequality gives

$$(2.13) \quad \begin{aligned} & |f(xy) + f(yx) - 2f(x) - 2f(y) + 2f(e)| \\ & \leq |f(xy) + f(x\sigma(y)) - 2f(x)| + |f(yx) + f(y\sigma(x)) - 2f(y)| \\ & \quad + |2f(e) - f(x\sigma(y)) - f(y\sigma(x))| \\ & \leq 3\delta. \end{aligned}$$

Hence, from (2.1) (2.12) and (2.13) we get

$$(2.14) \quad \begin{aligned} & |f(yx) + f(\sigma(y)x) - 2f(x)| \leq |f(yx) + f(xy) - 2f(y) - 2f(x) + 2f(e)| \\ & \quad + |f(\sigma(y)x) + f(x\sigma(y)) - 2f(\sigma(y)) - 2f(x) + 2f(e)| \\ & \quad + |-f(xy) - f(x\sigma(y)) + 2f(x)| + |2f(y) + 2f(\sigma(y)) - 4f(e)| \leq 9\delta \end{aligned}$$

Now, from (2.1) and (2.14) we obtain

$$(2.15) \quad \begin{aligned} & |f(yx) - f(\sigma(x)\sigma(y)) + f(y\sigma(x)) - f(x\sigma(y)) - 2(f(y) - f(\sigma(y)))| \\ & \leq |f(yx) + f(y\sigma(x)) - 2f(y)| + |f(x\sigma(y)) + f(\sigma(x)\sigma(y)) - 2f(\sigma(y))| \\ & \leq 10\delta. \end{aligned}$$

Consequently, we get

$$(2.16) \quad |f^o(yx) + f^o(y\sigma(x)) - 2f^o(y)| \leq 5\delta$$

for all $x, y \in G$. So, for fixed $y \in G$, the functions $x \mapsto f^o(yx) - f^o(x\sigma(y))$ and $x \mapsto f^o(xy) + f^o(x\sigma(y)) - 2f^o(x)$ are bounded on G . Furthermore,

$$(2.17) \quad \begin{aligned} m\{f_{\sigma(y)\sigma(z)}^o + f_{\sigma(y)z}^o - 2f_{\sigma(y)}^o\} &= m\{(f_{\sigma(z)}^o + f_z^o - 2f^o)_{\sigma(y)}\} \\ &= m\{f_{\sigma(z)}^o + f_z^o - 2f^o\}, \end{aligned}$$

where m is an invariant mean on G .

By using some computation as the one of (2.14) we get, for every fixed $y \in G$ the function $x \mapsto f^o(yx) + f^o(\sigma(y)x) - 2f^o(x)$ is bounded and

$$(2.18) \quad \begin{aligned} m\{_{zy}f^o + _{\sigma(z)y}f^o - 2_yf^o\} &= m\{_y(zf^o + _{\sigma(z)}f^o - 2f^o)\} \\ &= m\{_zf^o + _{\sigma(z)}f^o - 2f^o\} \end{aligned}$$

Now, define

$$(2.19) \quad \phi(y) := m\{_yf^o - f_{\sigma(y)}^o\}, \quad y \in G.$$

By using the definition of ϕ and m the equalities (2.17) and (2.18), we get

$$\begin{aligned}
 (2.20) \quad \phi(z y) + \phi(\sigma(z) y) &= m\{z y f^o - f_{\sigma(y)\sigma(z)}^o\} + m\{\sigma(z) y f^o - f_{\sigma(y)z}^o\} \\
 &= m\{z y f^o + \sigma(z) y f^o - 2 y f^o\} - m\{f_{\sigma(y)\sigma(z)}^o + f_{\sigma(y)z}^o - 2 f_{\sigma(y)}^o\} \\
 &\quad + 2m\{y f^o - f_{\sigma(y)}^o\} \\
 &= m\{z f^o + \sigma(z) f^o - 2 f^o\} - m\{f_{\sigma(z)}^o + f_z^o - 2 f^o\} \\
 &\quad + 2m\{y f^o - f_{\sigma(y)}^o\} \\
 &= m\{z f^o - f_{\sigma(z)}^o\} + m\{\sigma(z) f^o - f_z^o\} + 2m\{y f^o - f_{\sigma(y)}^o\} \\
 &= 2\phi(y) + \phi(z) + \phi(\sigma(z)),
 \end{aligned}$$

which implies that ϕ is a solution of the Drygas functional equation (2.3). Furthermore, we have

$$\begin{aligned}
 (2.21) \quad & \left| \frac{\phi}{2}(y) - f^o(y) \right| = \frac{1}{2} \left| \phi(y) - 2f^o(y) \right| = \frac{1}{2} \left| m\{y f^o - f_{\sigma(y)}^o - 2f^o(y)\} \right| \\
 & \leq \frac{1}{2} \left| m \right| \sup_{x \in G} \left| f^o(yx) - f^o(x\sigma(y)) - 2f^o(y) \right| = \frac{1}{2} \sup_{x \in G} \left| f^o(yx) + f^o(y\sigma(x)) - 2f^o(y) \right| \\
 & \leq \frac{5}{2} \delta.
 \end{aligned}$$

Now, by using Lemma 2.2: the mapping $\frac{\phi}{2}$ is a solution of Drygas functional equation (2.3) and $\frac{\phi}{2} - f^o$ is a bounded mapping, so we have

$$(2.22) \quad \frac{\phi}{2} = \lim_{n \rightarrow +\infty} 2^{-n} f^o(x^{2^n}),$$

which implies that $\frac{\phi}{2}$ is odd, so $\frac{\phi}{2}$ is a solution of Jensen functional equation (1.1). Consequently, we have

$$\begin{aligned}
 (2.23) \quad & \left| f(x) - \frac{\phi}{2} - f(e) \right| = \left| f^e(x) + f^o(x) - \frac{\phi}{2} - f(e) \right| \\
 & \leq \left| f^e(x) - f(e) \right| + \left| f^o(x) - \frac{\phi}{2} \right| \\
 & \leq \frac{\delta}{2} + \frac{5\delta}{2} = \frac{\delta}{2} = 3\delta.
 \end{aligned}$$

For proving the uniqueness of the obtained solution, we use the following: if g is a solution of equation (1.1), then

$$(2.24) \quad g(e) = g^e(x) = g(x\sigma(x)); \quad g(x^{2^n}) + (2^n - 1)g(e) = 2^n g(x)$$

for all $n \in \mathbb{N}$ and for all $x \in G$.

By using (2.24) and the proof of [Proposition 3, [18]] we get the following result.

Corollary 2.4. *Let G be an amenable semigroup with neutral element e and B a Banach space Let $f: G \longrightarrow B$ be a function satisfying the following inequality*

$$(2.25) \quad \| f(xy) + f(x\sigma(y)) - 2f(x) \| \leq \delta$$

for all $x, y \in G$ and for some nonnegative δ . Then there exists a unique solution g of the Jensen equation (1.1) such that

$$(2.26) \quad \| f(x) - g(x) - f(e) \| \leq 3\delta$$

for all $x \in G$.

Corollary 2.5. *Let G be an amenable semigroup with neutral element e and B a Banach space Let $f: G \longrightarrow B$ be a function satisfying the following inequality*

$$(2.27) \quad \| f(xy) + f(xy^{-1}) - 2f(x) \| \leq \delta$$

for all $x, y \in G$ and for some nonnegative δ . Then there exists a unique solution g of the Jensen equation (1.2) such that

$$(2.28) \quad \| f(x) - g(x) - f(e) \| \leq 3\delta$$

for all $x \in G$.

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Belaid Bouikhalene,
 Department of Mathematics,
 Polydisciplinary Faculty, Sultan Moulay Slimane University,
 Beni-Mellal, Morocco, E-mail: bbouikhalene@yahoo.fr

Elqorachi Elhoucien,
 Department of Mathematics, Faculty of Sciences, Ibn Zohr University,
 Agadir, Morocco, E-mail: elqorachi@hotmail.com